Rigidity of Configurations of Points and Spheres

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Abstract

The rigidity of collections of ideal points and collections of circles in two dimensions is proven by Beardon and Minda using a maximal amount of conformal invariant information. Crane and Short do the same for collections of ideal points and spheres in higher dimensions. In order to uniquely place collections of ideal points in $\mathbb{R}^n_\infty$, the cross ratio of every 4-tuple of points must be known. Similarly, rigidity of a collection of spheres in $\mathbb{R}^n_\infty$ uses the inversive distance between every pair of spheres. When these configurations are additionally required to have an independent subcollection, the amount of necessary conformal invariant information is considerably decreased by way of basic linear algebra in Lorentz space. We also consider ways to generalize to rigidity of collections of points and spheres together, touch upon the role of independence in rigidity of inversive distance circle packings, and its potential utility in rigidity of projective polyhedra.

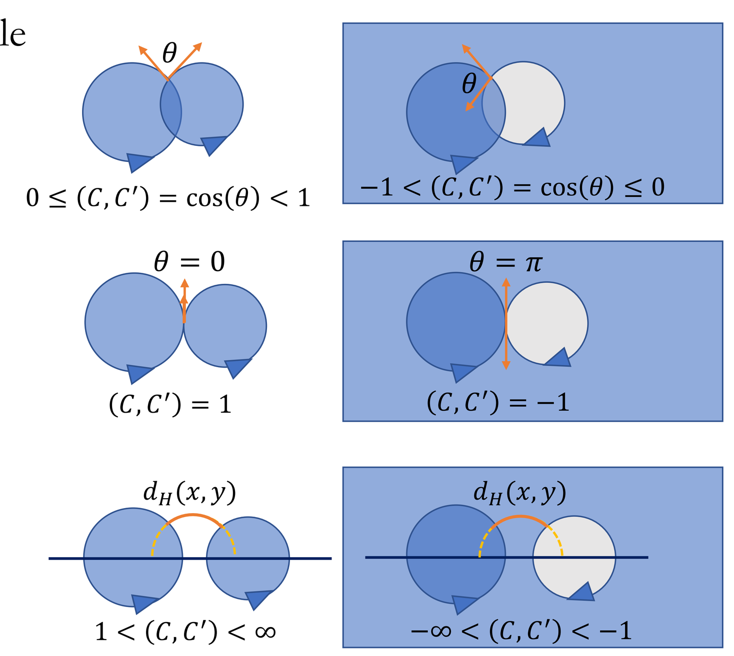
Conformal Invariants

Inversive distance (C, C') is a real number equipped to two circles (or spheres, in higher dimensions), C and C' (for more info, see [1]). It is…

- An invariant under Möbius transformations.
- Not a true distance:
  - ➢ Does not satisfy the triangle inequality
  - ➢ Can be negative.
- A tool commonly used in circle configurations.

The **Absolute cross ratio** $|a,b,c,d|$ is a real number equipped to four points a, b, c, d in $\mathbb{R}^N_\infty$, given by the formula $|a,b,c,d| = \frac{|a-b||c-d|}{|a-c||b-d|}$.

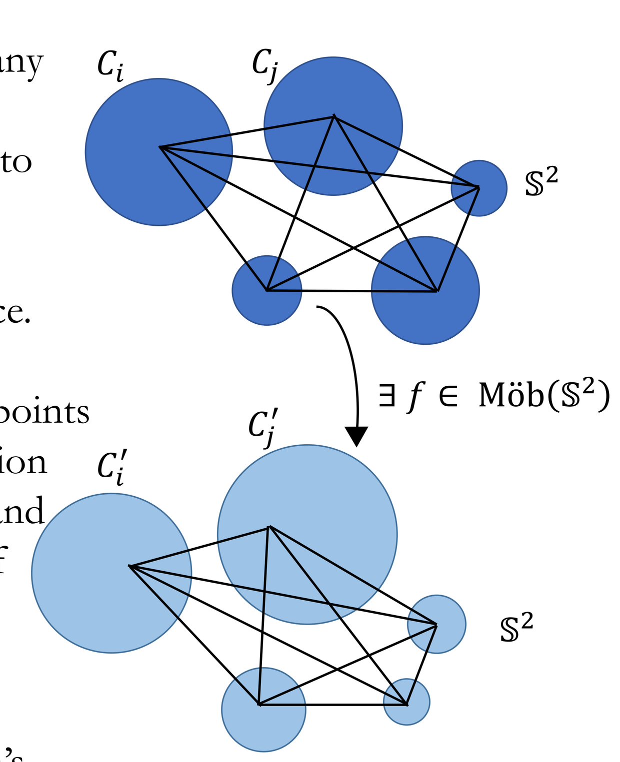
- It is also invariant under Möbius transformations.



Motivation

Beardon and Minda in [2] showed:

- In $\mathbb{C}_\infty$, two collections of finitely many disjoint disks have a Möbius transformation taking one collection to the other if and only if each corresponding pair of disks in the collections has equal inversive distance.
- For two collections of finitely many points in $\mathbb{C}_\infty$, there is a Möbius transformation taking one collection to the other if and only if each corresponding 4-tuple of points has equal absolute cross ratio.



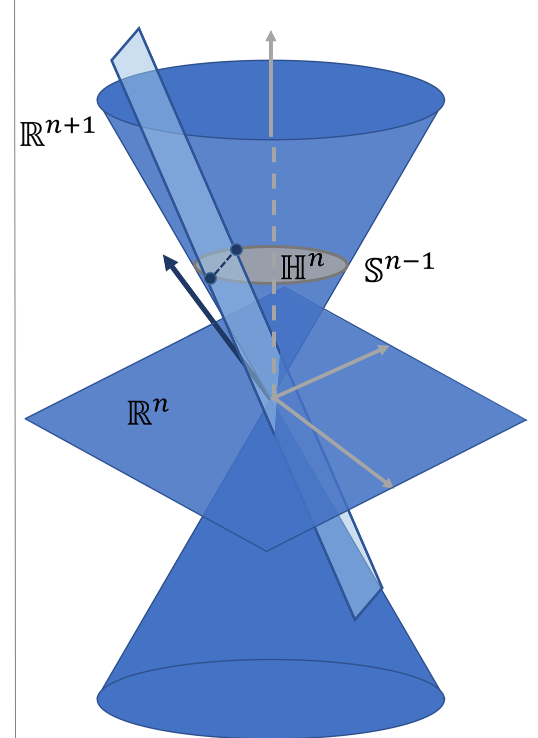
Crane and Short build on this in [3] by generalizing each of Beardon and Minda's results:

- $(N-1)$-spheres in $\mathbb{R}^N_\infty$ and points in $\mathbb{R}^N_\infty$.
- Spheres may intersect,
- Collections may be countably infinite.

Both sets of results utilize a maximal amount of conformal invariant information. Rigidity can be achieved with less conformal invariant information with an appropriate additional condition.

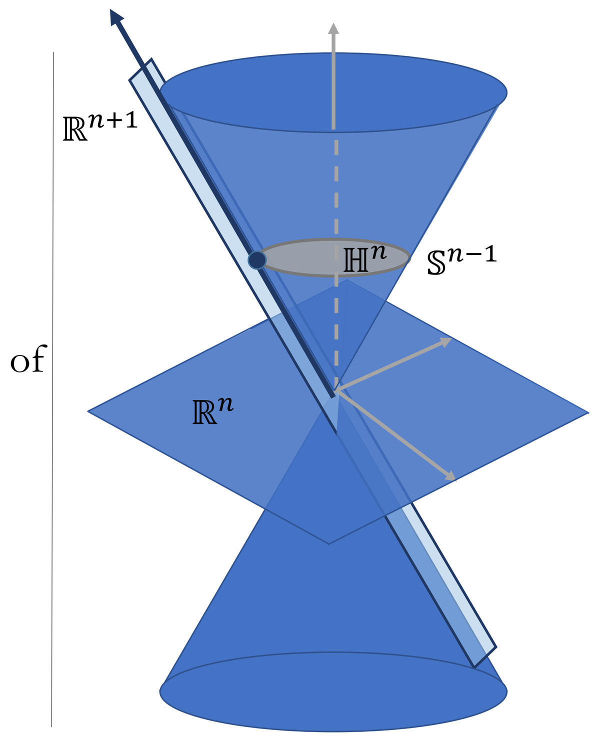
Lorentz Space Correspondences

- $\mathbb{R}^{n+1}$, together with the **Lorentz inner product** $\langle u, v\rangle = u_1v_1 + \cdots + u_nv_n - u_{n+1}v_{n+1}$, for two vectors u and v, is called **Lorentz space.** Lorentz subspaces are **space-like** if they don't intersect the light cone, **light-like** if they are tangent to the light cone, and **time-like** otherwise.
- **Möbius transformations in $\mathbb{S}^{n-1}$:** $SO^+(n,1) \cong \text{Isom}^+(\mathbb{H}^n) \cong \text{Möb}(\mathbb{S}^{n-1})$



- $(n-2)$-spheres in $\mathbb{S}^{n-1}$ ↔
- Time-like Lorentz subspaces of $\mathbb{R}^{n+1}$ ↔
- Space-like Lorentz vectors

Fact: The Lorentz inner product of two space-like vectors is equal to inversive distance of the corresponding $(n-2)$-spheres.

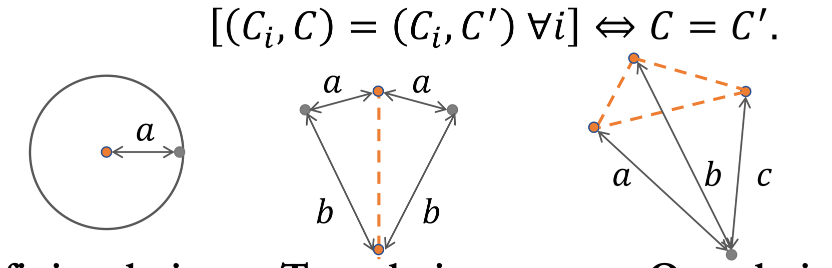


- Ideal points in $\mathbb{S}^{n-1}$ ↔
- Light-like Lorentz Subspaces ↔
- Light-like Lorentz vectors

Fact: For four ideal points a, b, c, d, $|a,b,c,d| = \frac{\langle v_a, v_b\rangle\langle v_c, v_d\rangle}{\langle v_a, v_c\rangle\langle v_b, v_d\rangle}$, where v_a, v_b, v_c, v_d are chosen light-like vectors intersecting their respective points.

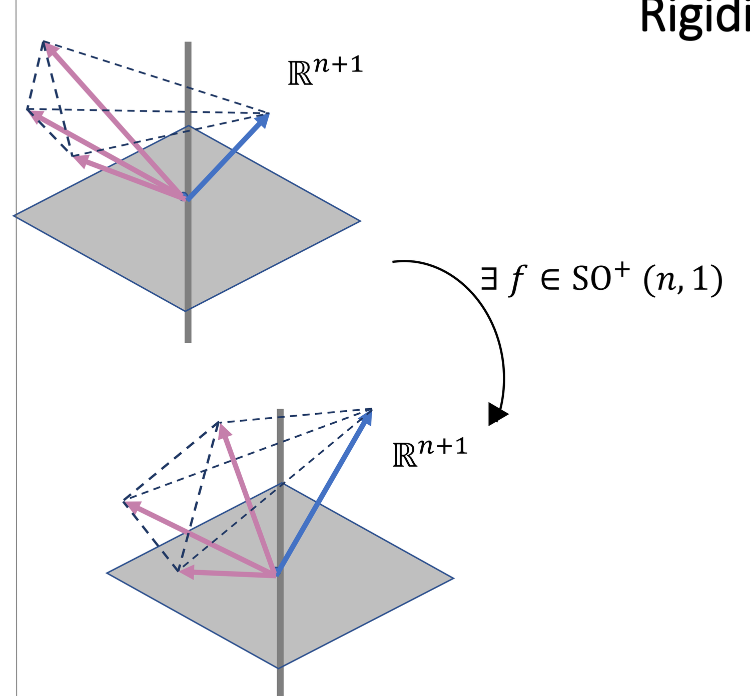
Independence of Circles

- A collection of $(N-2)$-spheres in $\mathbb{S}^{N-1} \subset \mathbb{R}^{N+1}$ is **independent** if their corresponding Lorentz vectors are linearly independent.
- **Lemma.** Let $\{C_1, \ldots, C_{n+1}\}$ be a collection of fixed independent spheres in $\mathbb{S}^{n-1} \subset \mathbb{R}^{n+1}$. For spheres C and C',
$$[(C_i, C) = (C_i, C')\ \forall i] \Leftrightarrow C = C'.$$
- **Analogous to:**



Infinite choices **Two choices** **One choice**

Rigidity of Configurations of Lorentz Vectors

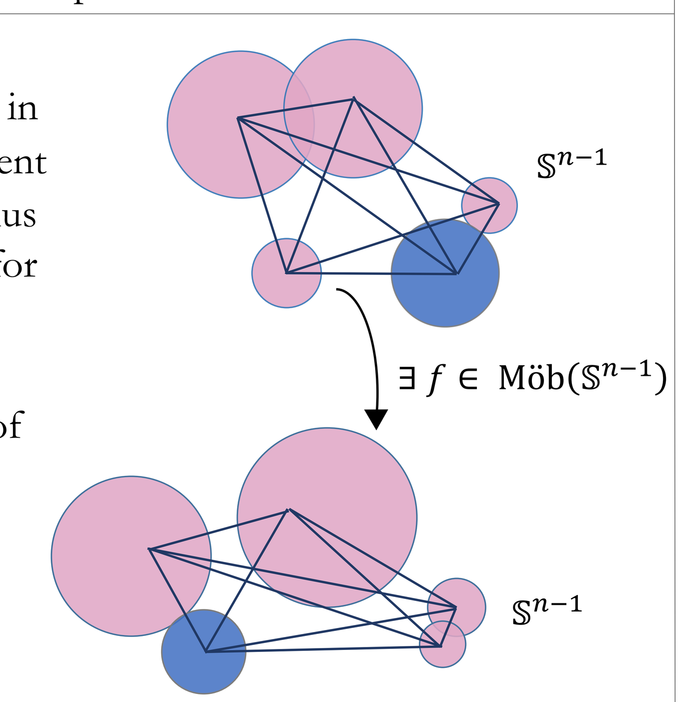


Theorem. Let $\{v_\alpha : \alpha \in \mathcal{A}\}$ and $\{v_\alpha' : \alpha \in \mathcal{A}\}$ be two collections of vectors in $\mathbb{R}^{n+1}$, indexed by the same set, with at least n+1 elements $\{v_i\}$ and $\{v_i'\}$, respectively, that form linearly independent sets. Then $\langle v_\alpha, v_i\rangle = \langle v_\alpha', v_i'\rangle$ for $i = 1, \ldots, n+1$ if and only if there is a Lorentz transformation f such that $f(v_\alpha) = v_\alpha'$ for each α in $\mathcal{A}$.

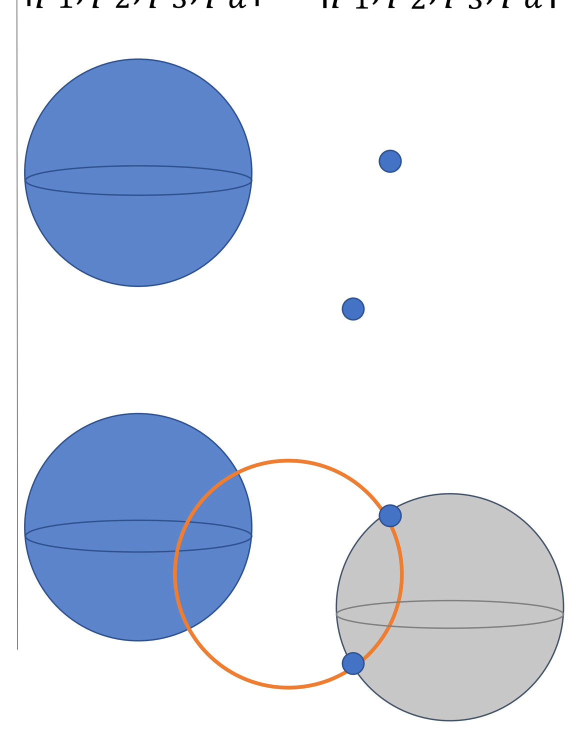
Remark. Independence greatly reduces the amount of necessary conformal invariant information to gain rigidity in each geometric interpretation.

Geometric Interpretations

Spheres. Two collections of $(n-2)$-spheres in $\mathbb{S}^{n-1} \subset \mathbb{R}^{n+1}$, $\{C_\alpha\}$ and $\{C_\alpha'\}$, with independent subcollections. $\{C_i\}_{i=1}^{n+1}$ and $\{C_i'\}_{i=1}^{n+1}$ are Möbius equivalent if and only if $(C_\alpha, C_i) = (C_\alpha', C_i')$ for $i = 1, \ldots, n+1$.



Points. Two collections $\{p_\alpha\}$ and $\{p_\alpha'\}$ of points in $\mathbb{R}^{n-1}_\infty$ with independent subcollections of n + 1 points have a Möbius transformation taking one to another if and only if $|p_1, p_2, p_3, p_\alpha| = |p_1', p_2', p_3', p_\alpha'|$ for chosen p_1, p_2, p_3 in the independent subcollection.



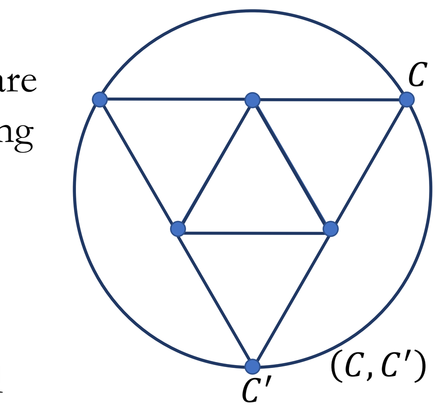
Points and Spheres. One possible approach is to transform collections of points and spheres into collections of spheres only:

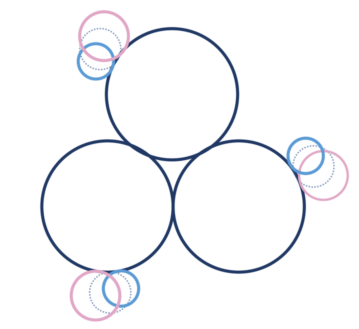
- For each pair of points in the collection, use a sphere in the collection to find a unique circle orthogonal to the sphere that passes through both the points.
- Use this circle to find a unique sphere orthogonal to the circle, intersecting the circle at the two points.
- If the newly formed collections of spheres satisfies the independent rigidity requirements above, then there is a Möbius transformation taking one *original* collection to the other.
- Note: Points in collection cannot lie on spheres in original collection.

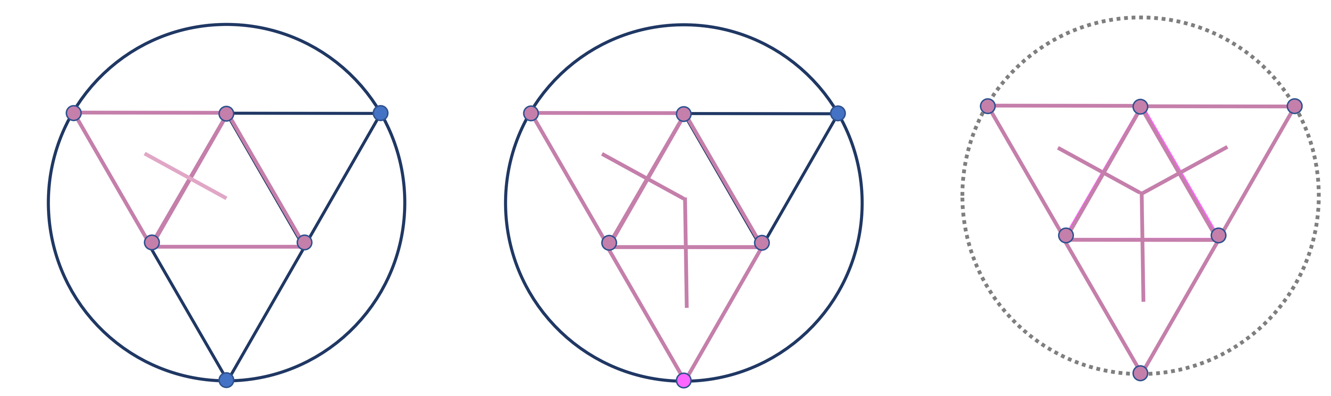
Future Research

1. Inversive Distance Circle Packings (IDCPs) are configurations of circles on $\mathbb{S}^2$ with an underlying triangulation encoding inversive distance information.

- IDCPs are not globally rigid in general:
  - ➢ Bowers and Bowers provide a counterexample in [5] using an octahedral graph.
  - ➢ By adding in extra inversive distance information strategically, independence can be used to "rigidify" the collection of circles.

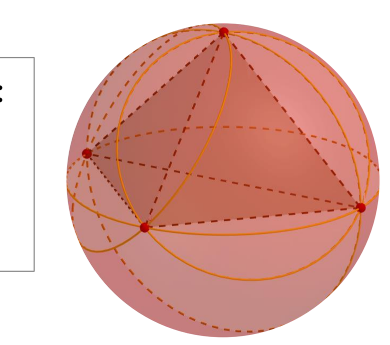




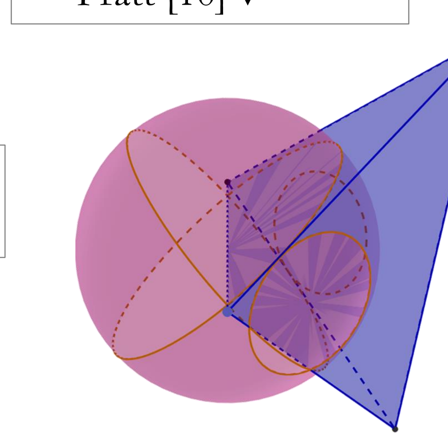


2. Projective polyhedra correspond to circle configurations in $\mathbb{S}^2$ called **circle polyhedra.** The rigidity of projective polyhedra up to Lorentz transformations is not yet fully known, although several cases have been classified (see below). Conditions involving independence imposed on circle polyhedra may be the right requirements to gain uniqueness, which will, in turn, yield uniqueness of projective polyhedra.

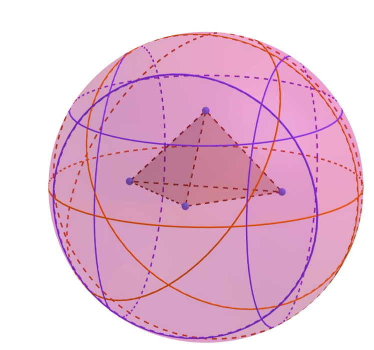
Hyperbolic polyhedra :
- Andre'ev [6]
- Rivin & Hodgson [7]▼



Hyperideal polyhedra :
- Bao & Bonahon [9]
- Bowers, Bowers, Pratt [10]▼



▲Ideal Polyhedra: Rivin [8]



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